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Stripline Monitors as Transmission Lines

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I. INTRODUCTION

Unlike the Tevatron, the proposed Fermilab Main Injector has only one beam circulating, either proton in one direction at one time or antiproton circulating in the other direction at another time. As a result, a beam-position monitor with one terminal per stripline will be enough. This can be accomplished by shorting one end of the directional beam-position monitor in the Tevatron.[1] The monitor impedances seen by the beam should be unchanged and so does the signal monitored at the terminal. The elimination of one terminal not only leads to simplification of monitor fabrication and installation. It amounts to a lot of savings also.

The computation of stripline impedances has been discussed by many authors.[2,3] However, most of the discussions have been more or less intuition oriented. We would like to bring in the concept of transmission line rigorously and see how a bunch current is reflected and transmitted. A time-domain picture is used and followed throughout, although some computations have to be done in the frequency domain when the concept of impedance is introduced. For simplicity, we assume the transmission line to be nondissipative and nondispersive. The equations governing the transmission line and the input impedance with various terminations are reviewed in the Appendix. We discuss in Sects. II and III, respectively, the stripline shorted at one end and the directional stripline with matched terminations at both ends. In Sect. IV, we derive the impedance and monitored signal of a stripline terminated at the center, and discover that there are no resonances when the velocity of the beam particles is equal to the velocity of the transmission line. In Sect. V, transverse impedances are discussed.

II. STRIPLINE SHORTED AT ONE END

This is the proposed stripline that we are interesting in. It consists of a plate of length ℓ with termination impedance Z_0 at the upstream end ($z=0$) and shorted at the downstream end ($z=\ell$) as shown in Fig. 1a. For the convenience of discussion, we extend the plate to $z=-\xi$ as in Fig. 1b and take the limit $\xi\to 0$ at the end. The stripline is at a constant distance from the beam pipe surface, forming with it a transmission line of characteristic impedance Z_0 and velocity v. Consider a localized ultra-relativistic bunch of current $i_0(t-z/c)$ defined by

$$i_0(t-z/c)=\int d\omega I_0(\omega)e^{j\omega(t-z/c)} \tag{2.1}$$

traveling to the right, where c is velocity of the beam particles which we take as the velocity of light. An image pulse current $-i_0(t-z/c)$ will be induced on the upper surface of the stripline. To maintain neutrality, there should also be an image current $+i_0(t-z/c)$ on the lower surface for $-\xi<z<0$, which we are concentrating on now. Let us assume that this current pulse passes $z=0$ at time $t=0$. When this current reaches $z=0$, it sees a termination impedance Z_0 and a transmission line of characteristic impedance Z_0 in parallel. The current therefore splits into two equal parts with one half $\frac{1}{2}i_0(t)$ flowing across the termination. The other half $\frac{1}{2}i_0(t-z/v)$ flows into the $z>0$ part of the stripline, but with velocity v, the velocity of the transmission line. When it reaches the downstream end $z=\ell$ at time ℓ/v, it is reflected. When the reflected current reaches the upstream end $z=0$ again at time $2\ell/v$, it sees a matched termination Z_0 and is completely absorbed without any more reflection according to Eq. (A.12). Therefore, the total current in the transmission line can be written as [Eqs. (A.9) and (A.12)],

$$i_\ell(t,z)=\frac{1}{2}\left\{i_0(t-z/v)+i_0[(t-\ell/v)+(z-\ell)/v]\right\} , \tag{2.2}$$

and the total current flowing across the termination is

$$i_T(t)=\frac{1}{2}\left[i_0(t)-i_0(t-2\ell/v)\right] . \tag{2.3}$$

The positive sign is chosen for the reflected current in Eq. (2.2) because the stripline is shorted at $z=\ell/2$. The total current at $z=0$ is

$$i_\ell(t,0)+i_T(t)=i_0(t) \tag{2.4}$$

equal to the incident current at $z=0-$ as required. To see that Eq. (2.2) is indeed correct, let us go to the frequency domain. The currents become

$$\tilde{i}_\ell(\omega,z)=\frac{1}{2}\,I_0(\omega)\left[e^{-jkz}+e^{jk(z-2\ell)}\right] , \tag{2.5}$$

$$\tilde{i}_T(\omega)=\frac{1}{2}\,I_0(\omega)\left[1-e^{-j2k\ell}\right] , \tag{2.6}$$

where $k=\omega/v$. Their ratio at $z=0$ is

$$\frac{\tilde{i}_\ell(\omega,0)}{\tilde{i}_T(\omega)}=\frac{1-e^{-j2k\ell}}{1-e^{-j2k\ell}}=\frac{1}{j\tan k\ell} , \tag{2.7}$$

which is just the ratio of the termination impedance Z_0 to the input impedance $jZ_0\tan k\ell$ [Eq. (A.14)] of a shorted line, and is just the ratio into which the current at $z=0-$ should split in the frequency domain.

The potential along the stripline is, from Eqs. (2.5), (A.3) and (A.4),

$$e_\ell(t,z)=\frac{1}{2}Z_0\left\{i_0(t-z/v)-i_0[(t-\ell/v)-(z-\ell)/v]\right\} , \tag{2.8}$$

which vanishes at $z = \ell$ as required. In the frequency domain, using Eqs. (2.5) and (A.8) or taking the Fourier transform of Eq. (2.8), the potential is

$$\tilde{e}_\ell(\omega, z) = \frac{1}{2} Z_0 I_0(\omega) \left[e^{-jkz} - e^{jk(z-2\ell)}\right] \ . \tag{2.9}$$

The potential at $z=0$ is

$$\tilde{e}_\ell(\omega, 0) = \frac{1}{2} Z_0 I_0(\omega) \left[1 - e^{-j2k\ell}\right] \ , \tag{2.10}$$

which can also be obtained by multiplying the current through the termination $\tilde{i}_T$ by the termination impedance Z_0. The longitudinal impedance seen by the particle bunch or the image current at $z = 0-$ is therefore

$$Z_\parallel = \frac{\tilde{e}_\ell(\omega, 0)}{I_0(\omega)} = \frac{1}{2} Z_0 \left[1 - e^{-j2k\ell}\right] \ . \tag{2.11}$$

This is in fact exactly the same as the parallel impedance of the termination impedance Z_0 and the impedance of the shorted transmission line $jZ_0 \tan k\ell$.

In the time domain, the potential at $z = 0$ is

$$e_\ell(t, 0) = \frac{1}{2} Z_0 \left[i_0(t) - i_0(t - 2\ell/v)\right] \ . \tag{2.12}$$

Therefore, if we monitor at the termination, we will see first half the bunch pulse followed at a time $2\ell/v$ later by the other half but with opposite sign.

In the above, we assume that the stripline wraps around the beam completely. However, if the stripline subtends an angle ϕ_0 at the beam pipe axis, all the currents i_ℓ and i_T should be multiplied by $\phi_0/2\pi$ because this is the fraction of the image current of the beam collected by the stripline. This applies also to the potential $\tilde{e}_\ell$ of Eqs. (2.8), (2.9), and (2.12). However, the average potential drop seen by the beam is only $(\phi_0/2\pi)\tilde{e}_\ell$ because only a fraction $\phi_0/2\pi$ of the image current crosses the gap and sees the potential drop. Therefore, the impedance of the stripline as seen by the beam in Eq. (2.11) will be reduced by $(\phi_0/2\pi)^2$.

III. DIRECTIONAL STRIPLINE TERMINATED AT BOTH ENDS

The directional stripline is shown in Fig. 2. Similar to the shorted one discussed in Sect. II, the image current pulse on the underside of the stripline splits into two equal parts at $z=0$, one half going through the termination of Z_0. The other half goes into the transmission line formed by the stripline and the beam pipe and is totally absorbed by the termination impedance Z_0 at $z = \ell$ without reflection.

The image current on the upper surface of the stripline travels with the same velocity as the particle bunch. When it reaches the $z = \ell$ gap at time ℓ/c, it goes into the underside of the stripline. Here, it splits into two equal parts, one half going across the downstream termination while the other half traveling backward to the left and is absorbed by the upstream termination without reflection. Thus the current on the underside of the stripline is

$$i_\ell(t,z) = \frac{1}{2}\{i_0(t - z/v) + i_0[(t-\ell/c) + (z-\ell)/v]\} . \tag{3.1}$$

The currents passing through the upstream (left-hand) and downstream (right-hand) terminations are respectively,

$$i_{T_u}(t) = \frac{1}{2}\{i_0(t) - i_0(t-\ell/c-\ell/v)\} , \tag{3.2}$$

$$i_{T_d}(t) = \frac{1}{2}\{i_0(t-\ell/v) - i_0(t-\ell/c)\} . \tag{3.3}$$

The potential along the transmission line is, according to Eq. (A.4),

$$e_\ell(t,z) = \frac{1}{2}\, Z_0\{i_0[t - z/v] - i_0[(t-\ell/c) + (z-\ell)/v]\} , \tag{3.4}$$

which gives at $z = 0$ and $z = \ell$ the correct potential at the upstream and downstream terminations:

$$e_{T_u}(t) = \frac{1}{2}\, Z_0\{i_0(t) - i_0(t-\ell/c-\ell/v)\} , \tag{3.5}$$

$$e_{T_d}(t) = \frac{1}{2}\, Z_0\{i_0(t - \ell/v) - i_0(t - \ell/c)\} . \tag{3.6}$$

We see that if the velocity of the particle beam c is equal to the characteristic velocity v of the transmission line, the potential at the downstream termination $e_{T_d}(t)$ vanishes. For this reason, the situation becomes exactly the same as for the shorted stripline discussed in Sect. II. In practice, the velocity of a transmission line is very near to c, unless there is so obstructions in the line to lower the velocity. The potential at the upstream termination $e_{T_u}(t)$ displays also a current beam pulse followed by another beam pulse of the opposite sign at a time $2\ell/v$ later.

To derive the impedance of the stripline, we need to resort to the frequency domain. The potential along the stripline or Eq. (3.4) becomes

$$\tilde{e}_\ell(\omega,z) = \frac{1}{2}\, Z_0 I_0(\omega) \left\{e^{-jkz} - e^{jkz} e^{-jk\ell(1+v/c)}\right\} . \tag{3.7}$$

The potentials at the upstream and downstream ends are, respectively,

$$\tilde{e}_{T_u}(\omega) = \frac{1}{2} Z_0 I_0(\omega) \left\{1 - e^{-jk\ell(1+v/c)}\right\} , \tag{3.8}$$

$$\tilde{e}_{T_d}(\omega) = \frac{1}{2} Z_0 I_0(\omega) \left\{e^{-jk\ell} - e^{-jk\ell v/c}\right\} , \tag{3.9}$$

which can also be obtained by Fourier transforming Eqs. (3.5) and (3.6). These potentials are seen respectively by currents $I_0(\omega)$ at the upstream end and $I_0(\omega)e^{-j\omega\ell/c}$ at the downstream end, contributing impedances

$$Z_{\|_u} = \frac{1}{2} Z_0 \left\{1 - e^{-jk\ell(1+v/c)}\right\} , \tag{3.10}$$

$$Z_{\|_d} = -\frac{1}{2} Z_0 \left\{e^{-jk\ell(1-v/c)} - 1\right\} . \tag{3.11}$$

The longitudinal impedance of the whole stripline is the sum of the two or

$$Z_{\|} = \frac{1}{2} Z_0 \left\{\left[1 - e^{-jk\ell(1+v/c)}\right] + \left[1 - e^{-jk\ell(1-v/c)}\right]\right\} , \tag{3.12}$$

which is exactly Eq. (2.10) if $v = c$. The above impedance can be checked against the power dissipation across the two terminations; i.e.,

$$\frac{1}{2} |I_0(\omega)|^2 \,\mathcal{R}e\, Z_{\|} = \frac{1}{2Z_0} \left\{|\tilde{e}_{T_u}|^2 + |\tilde{e}_{T_d}|^2\right\} . \tag{3.13}$$

IV. STRIPLINE TERMINATED AT THE CENTER

The stripline has its upstream end at $z = -\ell/2$, its downstream end at $z = \ell/2$, and is terminated at $z = 0$ by an impedance Z_T as shown in Fig. 3. It forms a transmission line with the beam pipe having characteristic impedance Z_0 and velocity v. Besides reflections at the upstream and downstream ends, currents will also be partly transmitted and reflected at the termination. An analysis in the time domain will be extremely complicated, because it involves the summation of infinite number of reflected and transmitted pulses. Here, we first solve the problem in the frequency domain and transform the results back to the time domain. After that, direct derivation in the time domain is pursued.

1. Impedance seen by the beam

The particle beam current $i_0[t-(z+\ell/2)/v]$ feeds the stripline by inducing currents

$$i_0\left[t - \frac{z+\ell/2}{v}\right] , \qquad i_0\left[\left(t - \frac{\ell}{v}\right) + \frac{z-\ell/2}{c}\right] \tag{4.1}$$

on the underside of the stripline, respectively, at the left and right ends. For the discussion below, let us simplify the problem by assuming the equality of the particle beam velocity c and the line velocity v. This assumption is later relaxed in Sect IV.3.

In the frequency domain, the currents on the upstream (left) and downstream (right) sides of the underside of the stripline are, respectively,

$$\tilde{i}_u(z) = I_0(\omega)\left\{e^{-jk(z+\ell/2)} + a\left[e^{-jk(z+\ell/2)} - e^{jk(z+\ell/2)}\right]\right\} , \tag{4.2}$$

and

$$\tilde{i}_d(z) = I_0(\omega)\left\{e^{jk(z-3\ell/2)} + b\left[e^{-jk(z-\ell/2)} - e^{jk(z-\ell/2)}\right]\right\} , \tag{4.3}$$

where $I_0(\omega)$ is the Fourier transform of the bunch current at $z = -\ell/2$. In the above, $\tilde{i}_u$, $\tilde{i}_d$, a, and b are functions of ω, although the ω-dependency has been suppressed for convenience. The expressions in the squared brackets have been written in such a way that they vanish at $z = -\ell/2$ or $z = \ell/2$, the correct reflection condition for open-circuit ends. The corresponding potentials on the upstream and downstream sides are

$$\tilde{e}_u(z) = Z_0 I_0(\omega)\left\{e^{-jk(z+\ell/2)} + a\left[e^{-jk(z+\ell/2)} - e^{jk(z+\ell/2)}\right]\right\} , \tag{4.4}$$

and

$$\tilde{e}_d(z) = Z_0 I_0(\omega)\left\{-e^{jk(z-3\ell/2)} + b\left[e^{-jk(z-\ell/2)} + e^{jk(z-\ell/2)}\right]\right\} . \tag{4.5}$$

These potentials have to be equal at $z = 0$, therefore

$$e^{-jk\ell/2} + a\left[e^{-jk\ell/2} + e^{jk\ell/2}\right] = -e^{-j3k\ell/2} + b\left[e^{jk\ell/2} + e^{-jk\ell/2}\right] . \tag{4.6}$$

The current through the termination is

$$\tilde{i}_T = \tilde{i}_u(0) - \tilde{i}_d(0) = I_0(\omega)\left\{e^{-jk\ell/2} - e^{-jk3\ell/2} + (a+b)\left[e^{-jk\ell/2} + e^{jk\ell/2}\right]\right\} . \tag{4.7}$$

which is equal to $\tilde{e}_u(0)/Z_T$, or

$$Z_0\left\{e^{-jk\ell/2} + a\left[e^{-jk\ell/2} - e^{jk\ell/2}\right]\right\} = Z_T\left\{e^{-jk\ell/2} - e^{-jk3\ell/2} + (a+b)\left[e^{-jk\ell/2} + e^{jk\ell/2}\right]\right\} . \tag{4.8}$$

With $\alpha = e^{-jk\ell}$ and $\beta = Z_T/Z_0$, Eqs. (4.7) and (4.8) can be rewritten as

$$\begin{pmatrix} 1+\alpha & -1-\alpha \\ 1+\alpha+\beta(1-\alpha) & \beta(1-\alpha) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\alpha(1+\alpha) \\ -\alpha+\beta\alpha(1-\alpha) \end{pmatrix} , \tag{4.9}$$

from which one obtains readily

$$a = -\frac{\alpha}{1+\alpha + 2\beta(1-\alpha)} \qquad \text{and} \qquad b = -\frac{\alpha[\alpha+2\beta(1-\alpha)]}{1+\alpha + 2\beta(1-\alpha)} \ . \tag{4.10}$$

The potentials at the upstream end ($z = -\ell/2$) and downstream end ($z = \ell/2$) are, respectively,

$$\tilde{e}_u(-\ell/2) = Z_0 I_0(\omega)(1+2a) = Z_0 I_0(\omega)\frac{(2\beta+1)(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ , \tag{4.11}$$

$$\tilde{e}_d(\ell/2) = Z_0 I_0(\omega)(-\alpha+2b) = Z_0 I_0(\omega)\frac{\alpha(2\beta-1)(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ . \tag{4.12}$$

The longitudinal impedance seen by the beam crossing the upstream gap is obtained by dividing $\tilde{e}_u(-\ell/2)$ by $I_0(\omega)$,

$$Z_{\|u} = Z_0\frac{(2\beta+1)(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ . \tag{4.13}$$

The longitudinal impedance seen by the beam crossing the downstream gap is obtained by dividing $-\tilde{e}_d(\ell/2)$ by $\alpha I_0(\omega)$, where α is the transit-time phase lag of the bunch,

$$Z_{\|d} = -Z_0\frac{(2\beta-1)(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ . \tag{4.14}$$

The total longitudinal impedance of the stripline is the sum of the two, or

$$Z_{\|} = Z_0\frac{2(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ . \tag{4.15}$$

Equation (4.15) is more complicated than Eq. (3.8) because there are infinite reflections and both ends and the center of the stripline. This remains true even if we let $Z_T = Z_0$. At low frequency, the stripline impedance becomes

$$Z_{\|} = jk\ell[1 - j\beta k\ell] \ , \tag{4.16}$$

which is inductive.

2. Potential at the terminal

Let us examine the potential at the termination ($z = 0$), which is

$$\tilde{e}_T = Z_0 I_0(\omega)e^{jk\ell/2}[\alpha + a(1+\alpha)] \ . \tag{4.17}$$

Substituting the solution for a, we get

$$\tilde{e}_T = Z_0 I_0(\omega)\alpha^{1/2}\frac{2\beta(1-\alpha)}{1+\alpha+2\beta(1-\alpha)} \ . \tag{4.18}$$

There is only power dissipation across the termination impedance, which is

$$P_T = \frac{|\tilde{e}_T|^2}{2Z_T} \ . \tag{4.19}$$

It is easy to verify with the aid of Eqs. (4.15) and (4.18) that this power lost is exactly equal to

$$P = \frac{1}{2}\,|I_0(\omega)|^2\,\mathcal{R}e\,Z_\parallel \ . \tag{4.20}$$

In order to study the temporal behavior, we expand Eq. (4.18) in powers of α:

$$\tilde{e}_T = \frac{1}{2}\,Z_0 I_0(1+\eta)\alpha^{1/2}[1-(1-\eta)\alpha-\eta(1-\eta)\alpha^2-\eta^2(1-\eta)\alpha^3-\eta^3(1-\eta)\alpha^4+\cdots] \ , \tag{4.21}$$

where

$$\eta = \frac{2\beta-1}{2\beta+1} \tag{4.22}$$

whose magnitude always less than unity. Transforming to the time domain, we have

$$\begin{aligned} e_T(t) = \frac{1}{2}\,Z_0(1+\eta)[i_0(t-\ell/2v) - (1-\eta)i_0(t-3\ell/2v) \\ -\eta(1-\eta)i_0(t-5\ell/2v)-\eta^2(1-\eta)i_0(t-7\ell/2v)-\eta^3(1-\eta)i_0(t-9\ell/2v)+\cdots] \ . \end{aligned} \tag{4.23}$$

The first term in the squared bracket represents the first arrival of the image pulse from the upstream end at time $\ell/2v$. Part of this pulse is reflected, part of it is transmitted, and the rest passes through the termination. At time ℓ/v, the reflected and transmitted parts reach, respectively, the upstream and downstream ends of the stripline and are reflected again. At this moment, the beam feeds the downstream end of the stripline with an image pulse [Eq. (4.1)]. All these arrive at the termination at time $3\ell/2v$ giving rise to the second term in the squared bracket of Eq. (4.23). The reflections and transmissions go on indefinitely.

3. Solution in the time domain

We now look into the time domain directly. The solution is illustrated in Fig. 4. We start with a beam pulse of current $i_0[t-(z+\ell/2)/v]$. This current enters the upstream (left) end at time zero. At a time $\ell/2v$ later, the current reaches the termination Z_T

$(z = 0)$. It is reflected as $\rho i_0[t + (z - \ell/2)/v]$ and transmitted as $\bar{\rho} i_0[t - (z + \ell/2)/v]$, where the reflection coefficient ρ and the transmission coefficient $\bar{\rho}$ are given according to Eq. (A.19) by

$$\rho = \frac{1}{2\beta + 1} = \frac{1-\eta}{2} \qquad \text{and} \qquad \bar{\rho} = 1 - \rho = \frac{2\beta}{2\beta + 1} = \frac{1+\eta}{2} \; , \tag{4.24}$$

with $\beta = Z_T/Z_0$. The current flowing down the termination is therefore

$$i_T = (1 + \rho - \bar{\rho}) i_0 \left[t - \ell/2v\right] \; . \tag{4.25}$$

The reflected and transmitted currents reach respectively the left and right ends at time ℓ/v. There, because the line is open, they are reflected as $-\rho i_0[t - (z + 3\ell/2)/v]$ and $-\bar{\rho} i_0[t + (z - 3\ell/2)/v]$. At this moment, the beam pulse reaches the downstream (right) end of the stripline and induces a current $i_0[t + (z - 3\ell/2)/v]$ flowing into the the stripline from the right. We continue to follow the reflections and transmissions of these two input sources. The reflected and transmitted currents are shown in Fig. 4. Those currents originate from the left-hand source are written above the arrows and those currents originate from the right-hand source are written below. The total current in a region of the stripline is the sum of all the reflected and transmitted currents in that region. For example, the current flowing through the termination at $z = 0$ is

$$\begin{aligned} i_T = {} & (1 + \rho - \bar{\rho}) i_0(t - \ell/2v) - (1 + \rho - \bar{\rho})^2 [i_0(t - 3\ell/2v) + (\bar{\rho} - \rho) i_0(t - 5\ell/2v) \\ & + (\bar{\rho} - \rho)^2 i_0(t - 7\ell/2v) + (\bar{\rho} - \rho)^3 i_0(t - 9\ell/2v) + \cdots] \; . \end{aligned} \tag{4.26}$$

The voltage across the termination is

$$e_T = Z_T i_T \; . \tag{4.27}$$

With the help of Eqs. (4.22) and (4.24), it is easy to show that Eq. (4.27) is indeed identical to Eq. (4.23).

4. Resonances

The longitudinal impedance of the stripline seen by the beam in Eq. (4.15) can be rewritten as

$$Z_{\|} = Z_0 (1 - \eta) \frac{1 - \alpha}{1 - \eta\alpha} \; , \tag{4.28}$$

where η is given by Eq. (4.22). For resonances to occur, we must have $\eta = \alpha^{-1} = e^{jk\ell}$ or

$$\beta = \frac{Z_T}{Z_0} = \frac{1}{2j \tan k\ell} \; . \tag{4.29}$$

The characteristic impedance Z_0 of a transmission line is mostly real, and so is the termination impedance Z_T, implying that resonances are not possible.

At the first thought, this result is very puzzling, because such of stripline system can support resonances. To demonstrate this, let us omit for the time being the current induced by the beam at the downstream port [Eq. (4.1)]. This is equivalent to omitting $e^{jk(z-3\ell/2)}$ from Eqs. (4.3) and (4.5), or $e^{-j3k\ell}$ from Eqs. (4.6) to (4.8), or the α^2 terms on the right side of Eq. (4.9). The result is

$$a' = -\frac{\alpha(1+\alpha-2\beta\alpha)}{(1+\alpha)[1+\alpha-2\beta(1-\alpha)]} \qquad \text{and} \qquad b' = -\frac{2\alpha\beta}{(1+\alpha)[1-\alpha+2\beta(1-\alpha)]} \ . \tag{4.30}$$

The potential at the upstream end $(z=-\ell/2)$ is

$$\tilde{e}'_u(-\ell/2) = Z_0 I_0(\omega)\frac{1+\eta\alpha^2}{(1+\alpha)(1-\eta\alpha)} \ , \tag{4.31}$$

which is just the input impedance of a transmission line of length $\ell/2$ terminated with an impedance formed from the parallel of Z_T and another open-ended transmission line of length $\ell/2$. The potential at the downstream end $(z=\ell/2)$ is

$$\tilde{e}'_d(\ell/2) = Z_0 I_0(\omega)\frac{(1-\eta)\alpha}{(1+\alpha)(1-\eta\alpha)} \ . \tag{4.32}$$

It is obvious from Eqs. (4.31) and (4.32) that the potential on the stripline resonates whenever

$$\alpha = e^{-jk\ell} = -1 \ , \tag{4.33}$$

or when the length of the stripline ℓ equals an odd number of half-wavelengths. In fact, the potential along the line for the lowest resonant mode ($\ell =$ half wavelength) is given by

$$\lim_{\alpha\to -1}(1+\alpha)\tilde{e}'(z) = -\sin\frac{\pi z}{\ell} \ . \tag{4.34}$$

Now let us include the induced current at the downstream port. Its effect to the stripline is exactly the same as the induced current at the upstream port. The only difference is that this downstream-port current is of negative sign going in the opposite direction and with a phase lag of $\alpha = e^{-jk\ell}$. It drives a resonance with potential exactly the same as Eq. (4.34), but with sign reversed. The resonance is therefore cancelled exactly.

More accurately, the transit-time phase lag of the bunch across the stripline should be

$$\alpha' = e^{-j\omega\ell/c} = e^{-jk\ell v/c} \ , \tag{4.35}$$

where c is the velocity of the bunch particles. Accordingly, the potentials at the upstream and downstream ends are, respectively,

$$\tilde{\epsilon}_u(-\ell/2) = \tilde{\epsilon}'_u(-\ell/2) - \alpha'\tilde{\epsilon}'_d(\ell/2) \ , \tag{4.36}$$

$$\tilde{\epsilon}_d(\ell/2) = \tilde{\epsilon}'_u(-\ell/2) - \frac{\tilde{\epsilon}'_d(\ell/2)}{\alpha'} \ . \tag{4.37}$$

If the velocity of the transmission line v is equal to the velocity of the particle bunch c, it is easy to verify that the resonance factor $(1+\alpha)$ in the denominators of Eqs. (4.36) and (4.37) are cancelled so that the resonances disappear. We then recover Eqs. (4.11) and (4.12). However, if v is different from c, there are no such cancellation. From Eqs. (4.36) and (4.37), the longitudinal impedance seen by the beam will become

$$Z_{\parallel} = Z_0(1-\eta)\frac{1-\alpha}{1-\eta\alpha} + Z_0(1+\eta)\frac{(\alpha-\alpha')(\alpha\alpha'-1)}{(1+\alpha)(1-\eta\alpha)\alpha'} \ . \tag{4.38}$$

In the above, the first term is the impedance given by Eq. (4.15) or Eq. (4.28) and the second term accounts for the difference in v and c which contributes to the resonances.

From Eq. (4.33), the frequency of the n-th resonance is

$$\omega_n = (2n-1)\frac{\pi v}{\ell} \ . \tag{4.39}$$

The strength of the resonance is determined by the shunt impedance over figure of merit, R/Q. The latter is related to the residue of $Z_{\parallel}$ at $\omega = \omega_n$ by

$$\lim_{\omega\to\omega_n}(\omega-\omega_n)Z_{\parallel} = -\frac{\omega_n}{2j}\left(\frac{R}{Q}\right)_n \ . \tag{4.40}$$

We obtain

$$\left(\frac{R}{Q}\right)_n = \frac{8Z_0}{(2n-1)\pi}\cos^2\frac{\pi v}{2c} \ , \tag{4.41}$$

which is independent of Z_T as expected. When v is sufficiently close to c, Eq. (4.41) reduces to

$$\left(\frac{R}{Q}\right)_n = \frac{2\pi Z_0}{(2n-1)}\left(\frac{v}{c}-1\right)^2 \ . \tag{4.42}$$

V. TRANSVERSE IMPEDANCE

To measure the transverse offset of the beam from the center of the beam pipe, we need two striplines on either side of the beam, each subtending an angle ϕ_0 at the pipe axis. To compute the transverse impedance, we place a dipole beam at the beam axis and calculate the the amount of image current induced on the striplines. The voltage across the upstream and downstream gaps of each stripline can be derived as in the above discussions. The longitudinal impedance of the dipole mode, $Z_{\|}^{1}$ can then be inferred by equating the power loss at the termination impedances. The transverse impedance (of the dipole mode) can then be obtained using the relation

$$Z_{\perp}(\omega) = \frac{c}{\omega} Z_{\|}^{1}(\omega) \; . \tag{5.1}$$

The derivation has been given in detail in Ref. 3. In general, the transverse impedance in the direction from one stripline to the other of the pair is related to the longitudinal impedance of a pair by

$$Z_{\perp}(\omega) = \frac{c}{\omega b^2} \left(\frac{4}{\phi_0} \right)^2 \sin^2 \frac{\phi_0}{2} Z_{\|}(\omega) \; , \tag{5.2}$$

where b is the radius of curvature of the striplines encircling the beam. The transverse impedance in the other transverse direction is zero.

APPENDIX

1. The telegrapher's equations

A transmission line consists of two conducting surfaces running in say the z-direction. One of the surfaces is grounded. At time t and position z on the other surface, there is a current $i(t, z)$ and a potential $e(t, z)$, which are *not* independent of each other. Since we are interested in a lossless and nondispersive line, the line carries only an inductance L per unit length and a capacitance C per unit length, which are frequency independent. Across a length dz, the potential drops by de due to the inductance Ldz, and the current changes by di due to the capacitance Cdz (see Fig. 4). The voltage and current are therefore related by

$$\frac{\partial e}{\partial z} = -L\frac{\partial i}{\partial t} \, ,$$
$$\frac{\partial i}{\partial z} = -C\frac{\partial e}{\partial t} \, , \tag{A.1}$$

which are called the telegrapher's equations. It is clear from Eq. (A.1) that $i(t, z)$ and $e(t, z)$ satisfy a wave equation with a velocity of

$$v = \frac{1}{\sqrt{LC}} \, . \tag{A.2}$$

The general solution is

$$i(t, z) = f(t - z/v) + g(t + z/v) \, , \tag{A.3}$$

representing a wave traveling to to the right and a wave traveling to the left. With the help of Eq. (A.1), the potential can be obtained easily as

$$e(t, z) = Z_0[f(t - z/v) - g(t - z/v)] \, , \tag{A.4}$$

where we have introduced the characteristic impedance of the transmission line,

$$Z_0 = \sqrt{\frac{L}{C}} \, . \tag{A.5}$$

Attention should to paid to the signs in Eqs. (A.3) and (A.4).

In the frequency domain, we have for example,

$$i(t \mp z/v) = \int d\omega \; \tilde{i}_{\mp}(\omega, z) e^{j\omega t} \, , \tag{A.6}$$

where

$$\tilde{i}_{\mp}(\omega, z) = I_{\mp}(\omega) e^{\mp jkz} \qquad k = \omega/v \ , \tag{A.7}$$

represents wave traveling to the right (left), and $I_{\mp}(\omega)$ is given by Eq. (2.1). Substitution into Eq. (A.1) gives

$$\tilde{e}_{\mp}(\omega, z) = \pm Z_0 \tilde{i}_{\mp}(\omega, z) \ . \tag{A.8}$$

2. Input Impedance

Consider a transmission line of length ℓ and terminated with an impedance Z_T as shown in Fig. 5. Consider an input current flowing in the positive z-direction. This current will be reflected at the end of the line ($z = \ell$). So the total current is, for one frequency,

$$\tilde{i}(\omega, z) = I_0(\omega) \left[e^{-jkz} + \rho e^{jk(z-2\ell)} \right] \ , \tag{A.9}$$

where $k = \omega/v$ and ρ is the reflection coefficient. The potential along the line is, according to Eq. (A.8),

$$\tilde{e}(\omega, z) = Z_0 I_0(\omega) \left[e^{-jkz} - \rho e^{jk(z-2\ell)} \right] \ . \tag{A.10}$$

The reflection coefficient ρ can be determined easily by matching the terminated impedance Z_T to the ratio of the potential and current at $z = \ell$, or

$$Z_T = \frac{\tilde{e}(\omega, \ell)}{\tilde{i}(\omega, \ell)} = Z_0 \frac{1 - \rho}{1 + \rho} \ . \tag{A.11}$$

The reflection coefficient for the current is

$$\rho = \frac{Z_0 - Z_T}{Z_0 + Z_T} \ . \tag{A.12}$$

The reflection coefficient for the potential will be the negative of this. The time-domain picture can be obtained by performing inverse Fourier transforms on Eqs. (A.9) and (A.10). We see that, providing that Z_0 and Z_T are frequency-independent, the reflected current or potential has exactly the same wave form as the incident current or potential. In order words, providing that Z_0 and Z_T are frequency-independent, the above formulation can be carried out in the time domain.

The input impedance is the ratio of the potential to the current at $z = 0$, or

$$Z_i = Z_0 \frac{1 - \rho e^{-j2k\ell}}{1 + \rho e^{-j2k\ell}} = Z_0 \frac{j Z_0 \tan k\ell + Z_T}{Z_0 + j Z_T \tan k\ell} \ . \tag{A.13}$$

Note that when $Z_T = Z_0$, $\rho = 0$ implying that there is no reflection at the termination. In order words, the current (or potential) flowing into the termination is totally absorbed. Under this situation, the input impedance Z_i is just the characteristic impedance Z_0 of the line.

If the transmission line is short-circuited at $z = \ell$, i.e., $Z_T = 0$, the input impedance is

$$Z_i = jZ_0 \tan k\ell \ , \tag{A.14}$$

which is purely reactive and can take on any value depending on the frequency ω and the length ℓ of the line.

3. Reflection and transmission

Consider two transmission lines having the same velocity v and characteristic impedance Z_0 joined together by a termination impedance Z_T at $z = 0$ as shown in Fig. 7. A wave coming from the left will be partly reflected and transmitted. The total currents on the left and the right of the termination are given by

$$\tilde{i}_u(\omega, z) = I_0(\omega)\left[e^{-jkz} + \rho e^{jkz}\right] \tag{A.15}$$

and

$$\tilde{i}_d(\omega, z) = I_0(\omega)\bar{\rho}e^{-jkz} \tag{A.16}$$

respectively, where ρ and $\bar{\rho}$ are the reflection and transmission coefficients. The voltage at the termination can be obtained from Eqs. (A.15) or (A.16) by applying Eq. (A.4), or

$$\tilde{e}_T(\omega) = Z_0 I_0(\omega)(1 - \rho) = Z_0 I_0(\omega)\bar{\rho} \ . \tag{A.17}$$

The current through the termination impedance Z_T is

$$\tilde{i}_T(\omega) = I_0(\omega)(1 + \rho - \bar{\rho}) \ , \tag{A.18}$$

which is also equal to $\tilde{e}_T/Z_T$. Therefore, we get

$$\rho = \frac{1}{2\beta + 1} \qquad \text{and} \qquad \bar{\rho} = \frac{2\beta}{2\beta + 1} \ , \tag{A.19}$$

with $\rho + \bar{\rho} = 1$ and $\beta = Z_T/Z_0$. Note that if Z_T and Z_0 are frequency-independent, ρ and $\bar{\rho}$ are are frequency-independent. If we perform an inverse Fourier transform on Eq. (A.15) or Eq. (A.16), we will find that the current or potential wave form does not change after reflection and transmission.

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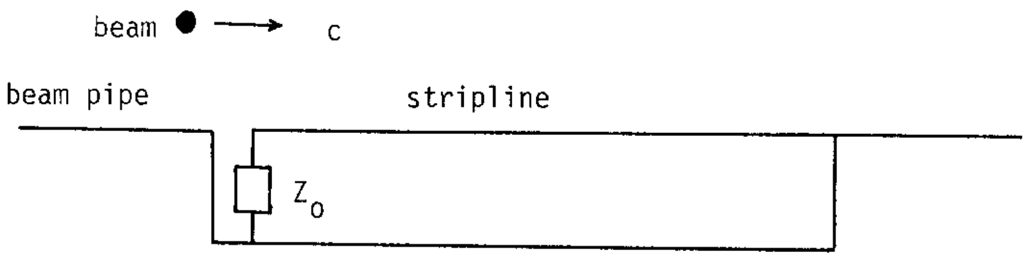


(a)

(b)

Fig. 1. Stripline terminated at near end and shorted at far end.

Fig. 2. Stripline terminated at both ends.

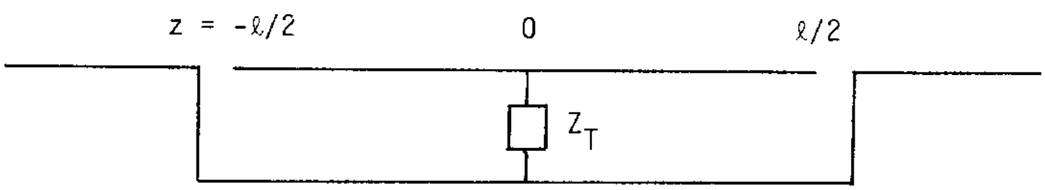


Fig. 3. Stripline terminated at the center.

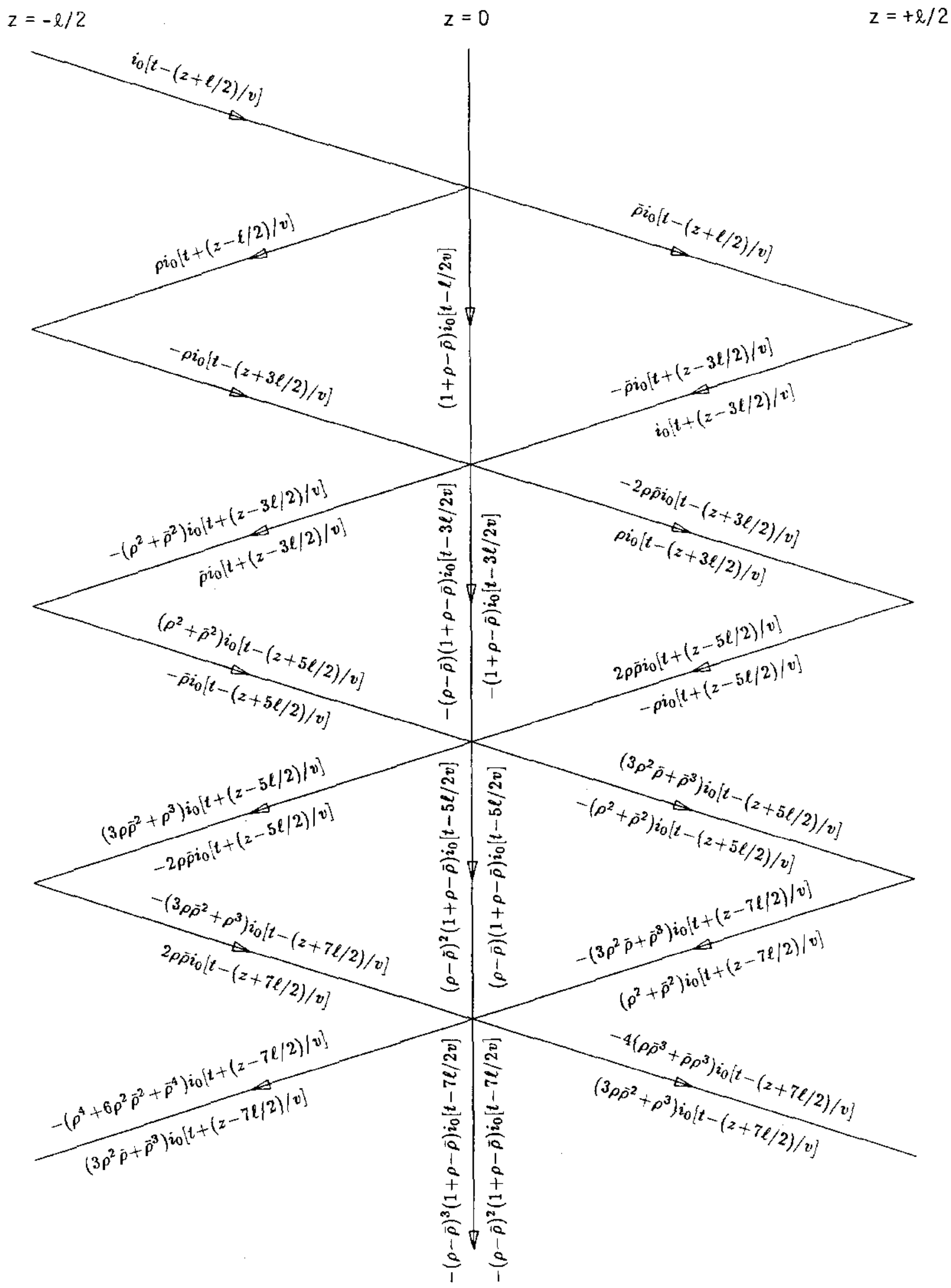
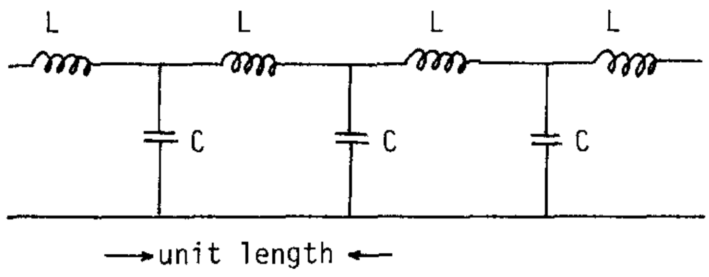


Fig. 4. Reflected and transmitted currents in a stripline terminated at the center. The source is a beam current $i_0[t-(z+\ell/2)/v]$.



Fiq. 5. Transmission line

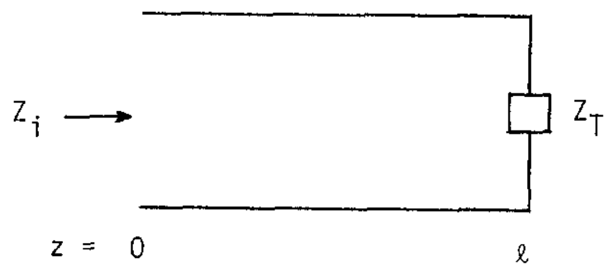


Fig. 6. Input impedance

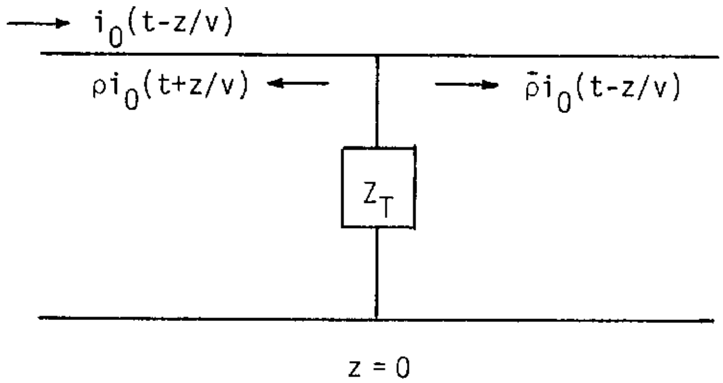


Fig. 7. Reflection and transmission at a termination. The reflection and transmission coefficients for current are denoted by ρ and $\bar{\rho}$.